Technical Note No. 90

DIPOLE AND QUADRUPOLE PLANAR MAGNETIC FIELDSPRODUCED BY LINE CURRENTS

by F. Deutsch*

1. INTRODUCTION

Dipole and quadrupole magnetic fields have important applications, in particular as lenses for charged-particle beams. In the following it will be shown that such fields can be obtained by means of line currents, and that with suitable geometries homogeneous dipole fields and constant gradient quadrupole fields can be produced.

2. VECTOR POTENTIAL

A line current of strength $+I$ is placed at the cylindrical coordinates r, φ . Then its vector potential A at a point $P(\rho, \psi)$ (Fig. 1) will be

$$A = -\frac{1}{2} \lambda \log [r^2 + \rho^2 - 2r\rho \cos(\varphi - \psi)] \quad (1)$$

with $\lambda = \frac{I\mu}{2\pi}$, μ permeability (constant and isotropic)

Fig. 1

The expression (1) for the vector potential becomes more manageable if transformed into trigonometric power series (see Appendix):

* Visitor from the Department of Physics, Hebrew University of Jerusalem, Israel

$$\frac{A}{\lambda} = -\frac{1}{2} \log [r^2 + \rho^2 - 2r\rho \cos(\varphi - \psi)] = \left(\sum_{k=1}^{\infty} \frac{a^k \cos k(\varphi - \psi)}{k} \right) - \log r \quad (1a)$$

with $a = \rho/r$ and valid for $a < 1$, i.e. $\rho < r$

and

$$\frac{A}{\lambda} = -\log \rho + \sum_{k=1}^{\infty} \frac{\cos k(\varphi - \psi)}{k a^k} \quad (1b)$$

for $a > 1$, i.e. $\rho > r$

Similar expressions can be obtained for the vector potential A in the point $P(\rho, \psi)$ from line currents in the 2nd, 3rd, and 4th quadrant. With φ designing the angle between the radius-vector and the x-axis (Fig. 2), and n as the number of the respective quadrant, we have for the vector potential from one line current placed in one of the four quadrants :

$$\frac{A}{\lambda} = -\log r + \sum_{k=1}^{\infty} \left\{ \left[-(-1)^{\frac{(n+1)(n+2)}{2}} \right]^k \frac{a^k \cos k[\varphi + (-1)^n \psi]}{k} \right\} \quad (2a)$$

for $a < 1$

and

$$\frac{A}{\lambda} = -\log \rho + \sum_{k=1}^{\infty} \left\{ \left[-(-1)^{\frac{(n+1)(n+2)}{2}} \right]^k \frac{\cos k[\varphi + (-1)^n \psi]}{k a^k} \right\} \quad (2b)$$

for $a > 1$

Fig. 2

It can be seen that for a pair of currents (+I and -I) $\log r$ and $\log \rho$ in (2a) and (2b) cancel out and only the sums remain.

It is now possible to build up various multi-line configurations giving dipole or quadrupole magnetic fields.

3. DIPOLE FIELDS

The vector potential in P (ρ, ψ) resulting from 4 line currents (Fig. 3) is

$$\frac{A}{\lambda} = \sum_{k=1}^{\infty} \frac{a^k \cos k(\varphi - \psi)}{k} + \sum_{k=1}^{\infty} (-1)^k \frac{a^k \cos k(\varphi + \psi)}{k} - \sum_{k=1}^{\infty} \left(\frac{a}{-1}\right)^k \frac{a^k \cos k(\varphi - \psi)}{k} - \sum_{k=1}^{\infty} \frac{a^k \cos k(\varphi + \psi)}{k} \quad (3)$$

Fig. 3

The series in (3) can in turn be summed, giving

$$\frac{A}{\lambda} = 4 \sum_{k=1}^{\infty} \frac{a^{2k-1}}{2k-1} \sin [(2k-1)\varphi] \sin [(2k-1)\psi] \quad (4)$$

for $a = \rho/r < 1$, and a similar expression for $a > 1$.

From (4) the ρ and ψ components of the magnetic field at P can be calculated :

$$B_{\rho} = (\text{rot } A)_{\rho} = \frac{1}{\rho} \frac{\partial A}{\partial \psi} = \frac{4\lambda}{r} \sum_{k=1}^{\infty} \frac{\rho}{r^{2k-2}} \sin(2k-1)\varphi \cos(2k-1)\psi \quad (5a)$$

$$B_{\psi} = (\text{rot } A)_{\psi} = -\frac{\partial A}{\partial \rho} = 4\lambda \frac{1}{r} \sum_{k=1}^{\infty} \frac{\rho}{r^{2k-2}} \sin(2k-1)\varphi \sin(2k-1)\psi \quad (5b)$$

for $\psi = 0$ (x-axis), $B_{\psi} = 0$, and

$$(B_\rho)_{\psi=0} = B_x = 4 \lambda \frac{1}{r} \sum_{k=1}^{\infty} \left(\frac{x}{r}\right)^{2k-2} \sin(2k-1) \varphi \quad (6a)$$

Similarly, for $\psi = 90^\circ$ (y-axis), $B_\rho = 0$, and

$$(B_\psi)_{\psi=90^\circ} = B_y = + 4 \lambda \frac{1}{r} \sum_{k=1}^{\infty} (-1)^k \left(\frac{y}{r}\right)^{2k-2} \sin(2k-1) \varphi \quad (6b)$$

It can be seen from (6a) and (6b) that at $\varphi = \pi/3$, the coefficient of x^2 vanishes. The homogeneity of such a field, produced from two pairs of line currents disposed on the vertices of a rectangle having $\frac{h}{a} = \sqrt{3} = \tan 60^\circ$ (Fig.3), is the best one can obtain from only four conductors.

Fig. 4 shows the deviation of the value of the magnetic field on the x and y-axis from its value at the origin.

The homogeneity of the dipole field can be improved by taking four pairs of line currents instead of two and disposing them symmetrically about the x-axis (Fig. 5).

The magnetic field resulting from this arrangement is

$$B_x = 4 \lambda \frac{1}{r} \sum_{k=1}^{\infty} \left(\frac{x}{r}\right)^{2k-2} \left[\sin (2k-1)\phi_1 + \sin (2k-1)\phi_2 \right] \quad (7a)$$

$$B_y = 4 \lambda \frac{1}{r} \sum_{k=1}^{\infty} (1)^k \left(\frac{y}{r}\right)^{2k-2} \left[\sin (2k-1)\phi_1 + \sin (2k-1)\phi_2 \right] \quad (7b)$$

The coefficients of both x^2 and x^4 vanish if

$$\begin{aligned} \sin 3\phi_1 + \sin 3\phi_2 &= 0 \\ \sin 5\phi_1 + \sin 5\phi_2 &= 0 \end{aligned} \quad (8)$$

The values $\phi_1 = 42^\circ$ and $\phi_2 = 78^\circ$ are the solution of (8). With these values (7a) and (7b) can be written as

$$B_x = 4 \lambda \frac{1}{r} 1,647 \left[1 - 0,621 \left(\frac{x}{r}\right)^6 + \left(\frac{x}{r}\right)^{10} - 0,621 \left(\frac{x}{r}\right)^{12} + \dots \right] \quad (9a)$$

$$- B_y = 4 \lambda \frac{1}{r} 1,647 \left[1 + 0,621 \left(\frac{x}{r}\right)^6 - \left(\frac{x}{r}\right)^{10} - 0,621 \left(\frac{x}{r}\right)^{12} + \dots \right] \quad (9b)$$

Fig. 6 shows the geometry giving a field according to expressions (9a) and (9b). The homogeneity of this field produced by 4 pairs of line currents is quite good for such a simple geometry. Fig. 7 shows the deviation from homogeneity along the x - and y - axes.

Fig 6

Fig 7

If the currents in the various pairs of wires can be different, we can find other interesting configurations giving fields of good homogeneity. For instance, the geometry of 3 pairs of line currents shown in Fig. 8 gives a field strength at any point x on the horizontal axis

$$B = \frac{2I\mu}{\pi} \left(1 - \frac{x^6}{r^6} + \frac{x^8}{r^8} \dots \right) \quad (10)$$

The deviation from homogeneity is only slightly greater than of the geometry of 4 pairs of line currents (Fig. 6), and reaches, for instance, 1.4% for 50% of the aperture and 0.35% for 40% of the aperture.

Fig.8

4. QUADRUPOLE FIELDS

Considered is a symmetrical arrangement of four line currents having alternating directions (Fig. 9)

Fig.9

The vector potential A for the four currents at the point P (ρ, ψ) can be obtained from (1a) and 1b). The result is

$$A = 4 \lambda \sum_{k=1}^{\infty} \frac{1}{2k} \left(\frac{\rho}{r}\right)^{2k} \sin 2k\varphi \sin 2k\psi \quad (\text{valid for } \rho < r) \quad (11a)$$

and

$$A = 4 \lambda \sum_{k=1}^{\infty} \frac{1}{2k} \left(\frac{r}{\rho}\right) \sin 2k\varphi \sin 2k\psi \quad (\text{valid for } \rho > r) \quad (11b)$$

From (11a) and 11b) we can calculate the components of the field strength at any point P (ρ, ψ) within the aperture :

$$B_{\rho} = (\text{rot } A)_{\rho} = \frac{1}{r} \frac{\partial A}{\partial \psi} = 4 \lambda \sum_{k=1}^{\infty} \frac{\rho^{2k-1}}{2k} \sin 2k\varphi \cos 2k\psi \quad (12a)$$

$$B_{\psi} = (\text{rot } A)_{\psi} = - \frac{\partial A}{\partial \rho} = - 4 \lambda \sum_{k=1}^{\infty} \frac{\rho^{2k-1}}{r^{2k}} \sin 2k\varphi \sin 2k\psi \quad (12b)$$

For $\varphi = 45^\circ$ and $\psi = 0$ (writing $p = x$) one obtains the expression

$$B_x = - \frac{4\lambda}{r^2} x \sum_{k=1}^{\infty} (-1)^k \left(\frac{x}{r}\right)^{4k-4} \quad (13)$$

in agreement with equation (18) of CERN 60-6, and similarly for the gradient

$$K(x) = \frac{dB_x}{dx} = - \frac{4\lambda}{r^2} \sum_{k=1}^{\infty} (-1)^k (4k-3) \left(\frac{x}{r}\right)^{4k-4} \quad (14)$$

in agreement with (20) in CERN 60-6.

We consider now gradients of the field produced by 8 line currents (see Fig. 10 on next page)

From 12a we can calculate the gradient for $\psi = 0$ (x-axis) :

$$(K_x)_{\psi=0} = \frac{4\lambda}{r^2} \sum_{k=1}^{\infty} \left(\frac{x}{r}\right)^{2k-2} (2k-1) \left[\sin 2k\phi_1 + \sin 2k\phi_2 \right] \quad (15)$$

We introduce now an additional condition : $\phi_1 = \pi/4 + \alpha$ and $\phi_2 = \pi/4 - \alpha$ (symmetry about the x - axis).

Expression (15) is changed with the above mentioned values for ϕ_1 and ϕ_2 into

$$K_x = - \frac{8\lambda}{r^2} \sum_{k=1}^{\infty} (4k-3) (-1)^k \cdot \cos (4k-2) \alpha \cdot \left(\frac{x}{r}\right)^{4k-4} \quad (16)$$

For $k = 1$ the gradient $K(0)$ at the origin is obtained. It can be seen that the coefficient of the next term (for $k = 2$) is zero if $\cos 6\alpha = 0$, i.e. $\alpha = 15^\circ$. With this value for α we obtain the following series for K_x :

$$K_x = \frac{4\sqrt{3}\lambda}{r^2} \left[1 - 9 \left(\frac{x}{r}\right)^8 + 13 \left(\frac{x}{r}\right)^{12} + \dots \right] \quad (17)$$

The relative error for the gradient is smaller than $- 9 \left(\frac{x}{r}\right)^8$.

For the gradient along the X - axis we obtain by a similar calculation

$$K(X) = - \frac{4\sqrt{3}\lambda}{r^2} \left[1 - 9 \left(\frac{x}{r}\right)^8 - 13 \left(\frac{x}{r}\right)^{12} \dots \right] \quad (18)$$

It is seen that the gradient errors are essentially the same in both x and X directions (as $13 \left(\frac{x}{r}\right)^{12}$ is extremely small for, say, $x < 0.6 r$), which was not the case for the gradients of the four line-current geometry.

The gradients for the two cases are compared in Fig. 11. The curves a) are for the case of the 4-line quadrupole, while curve b) represents the gradient of the 8-line quadrupole.

The gradient of the magnetic field has a still smaller error if a geometry of 16 wires, disposed symmetrically about the origin, is used (see Fig. 12 next page) :

Fig. 12

The gradient along the x-axis of this geometry is given by

$$K_x = -\frac{8\lambda}{r^2} \sum_{k=1}^{\infty} (-1)^k \left(\frac{x}{r}\right)^{4k-4} (4k-3) \left[\cos(4k-2) \alpha_1 + \cos(4k-2) \alpha_2 \right] \quad (19)$$

The coefficients of $\left(\frac{x}{r}\right)^4$ and $\left(\frac{x}{r}\right)^8$ (second and third terms of the sum) vanish if

$$\left. \begin{aligned} \cos 6 \alpha_1 + \cos 6 \alpha_2 &= 0 \\ \cos 10 \alpha_1 + \cos 10 \alpha_2 &= 0 \end{aligned} \right\} \quad (20)$$

The solutions of equation (20) are $\alpha_2 = 42^\circ$ and $\alpha_1 = 12^\circ$. With these values we can write for the gradient along the x-axis

$$K_x = \frac{8.144\lambda}{r^2} \left[1 + 21.0 \left(\frac{x}{4}\right)^{12} + \dots \right] \quad (21)$$

Fig. 13 shows the relative error of the gradient as a function of $x/4$ (curve a). The relative error of the 8-line currents geometry (equation (18)) is also shown, for comparison (curve b).

If wires of finite cross section are used, instead of line currents, then expressions (19) and (21) give only approximate values. However, it is easy to start from expressions (12a) and (12b) for the quadrupolar field from 4 line currents, divide the wires cross section into a great number of squares, and compute the sum of contributions from each elementary square.

Such a calculation is performed for 4 x 4 wires with $r_0/r = 0,05$; r_0 = wire radius; r radius of circle on which the wires are placed. The results are compared with those for 4 x 4 line currents in the following table :

$\frac{x}{r}$, $\frac{X}{r}$		0,2	0,3	0,4	0,5	0,6	Gradient at origin for 1 Amp. in each wire : $r=5$ cm
16 line currents	Δx	$6,68 \times 10^{-5}$	0,012	0,353	5,158	46,7	$6,516 \cdot 10^{-2}$
	ΔX	$-9,10 \times 10^{-5}$	-0,012	-0,353	-5,153	-46,3	
16 wires; aperture to wire diameter ratio: 20	Δx	$-9,12 \times 10^{-2}$	-0,252	-0,280	3,76	43,2	$6,517 \cdot 10^{-2}$
	ΔX	$1,30 \times 10^{-2}$	0,538	-0,171	-4,84	-46,3	

Δx , ΔX deviation in % from the gradient at origin.

5. ANNULAR QUADRUPOLES

In the following the vector potential A and field strength B (i.e. the components B_ρ and B_ψ) in any point P (ρ, ψ) shall be derived. The coil arrangement and symbols are given in Fig. 14 :

Fig. 14

In the preceding paragraphs it was found that the vector potential from 4 line currents in r, φ , in any point $P(\rho, \psi)$, when $r > \rho$ is :

$$A = 4\lambda \sum_{k=1}^{\infty} \frac{1}{2k} \left(\frac{\rho}{r}\right)^{2k} \sin 2k\varphi \sin 2k\psi \quad (22)$$

and when $r < \rho$:

$$A = 4\lambda \sum_{k=1}^{\infty} \frac{1}{2k} \left(\frac{r}{\rho}\right)^{2k} \sin 2k\varphi \sin 2k\psi \quad (23)$$

From (22) we can calculate the vector potential in the point P from all four coils :

$$A = \int_{r=R_1}^{R_2} \int_{\varphi=-\infty}^{(\pi/2)+\infty} \left(4\lambda \sum_{k=1}^{\infty} \left(\frac{\rho}{r}\right)^{2k} \frac{1}{2k} \sin 2k\varphi \sin 2k\psi \right) r dr d\varphi.$$

From this calculation we obtain :

$$\frac{A}{4\lambda} = \frac{\sin 2\psi \cos 2\alpha}{2} \rho^2 \ln \frac{R_2}{R_1} + \sum_{k=1}^{\infty} \frac{\sin(4k+2)\psi \cos(4k+2)\alpha}{2k(4k+2)^2} \rho^{4k+2} \cdot \left[\frac{1}{R_1^{4k}} - \frac{1}{R_2^{4k}} \right] \quad (24)$$

From (24) we can, in turn, obtain the components B_ρ and B_ψ of the magnetic field inside the aperture ($\rho < R_1$).

$$B_\rho = (\text{rot } A)_\rho = \frac{1}{\rho} \frac{\partial A}{\partial \psi} = 4\lambda \left\{ \cos 2\psi \cos 2\alpha \rho \ln \frac{R_2}{R_1} + \sum_{k=1}^{\infty} \frac{\cos(4k+2)\psi \cos(4k+2)\alpha}{2k(4k+2)} \rho^{4k+1} \left[\frac{1}{R_1^{4k}} - \frac{1}{R_2^{4k}} \right] \right\} \quad (25)$$

$$B_\psi = (\text{rot } A)_\psi = -\frac{\partial A}{\partial \rho} = -4\lambda \left\{ \sin 2\psi \cos 2\alpha \rho \ln \frac{R_2}{R_1} - \sum_{k=1}^{\infty} \frac{\sin(4k+2)\psi \cos(4k+2)\alpha}{2k(4k+2)} \rho^{4k+1} \left[\frac{1}{R_1^{4k}} - \frac{1}{R_2^{4k}} \right] \right\} \quad (26)$$

By substituting $\psi=0$ in (25) and $\psi=\pi/4$ in (26) we obtain the field strength along the main quadrupole axes; a further differentiation gives the gradients. The thus obtained expressions are identical with those derived by a different method in Report MPS/Int. MA 65-12.

Similarly we calculate the vector potential and field strength outside the aperture ($\rho > R_2$) (from 23) :

$$A = 4\lambda \left\{ \sum_{k=1}^{\infty} \frac{\sin(4k-2)\psi \cos(4k-2)\alpha}{2k(4k-2)^2}, \frac{1}{\rho^{4k-2}} \left(R_2^{4k} - R_1^{4k} \right) \right\} \quad (27)$$

and from this :

$$B_{\rho} = \frac{1}{\rho} \frac{\partial A}{\partial \psi} = 4\lambda \left\{ \sum_{k=1}^{\infty} \frac{\cos(4k-2)\psi \cos(4k-2)\alpha}{2k(4k-2)} \frac{1}{\rho^{4k-1}} \left(R_2^{4k} - R_1^{4k} \right) \right\} \quad (28)$$

and

$$B_{\psi} = - \frac{\partial A}{\partial \rho} = 4\lambda \left\{ \sum_{k=1}^{\infty} \frac{\sin(4k-2)\psi \cos(4k-2)\alpha}{2k(4k-2)} \frac{1}{\rho^{4k-1}} \left[R_2^{4k} - R_1^{4k} \right] \right\} \quad (29)$$

The knowledge of the field strength inside the coil is important for superconductivity coils. There the field strength inside the coil should not surpass a certain critical value, above which superconductivity is destroyed. For this purpose we have to know the field strength in any point for which $R_1 < \rho < R_2$. Using the expressions (17), (18), (20) and (21) we obtain for this region :

$$\begin{aligned} \frac{B_{\rho}}{4\lambda} &= \cos 2\psi \cos 2\alpha \rho \ln \frac{R_2}{\rho} + \sum_{k=1}^{\infty} \frac{\cos(4k+2)\psi \cos(4k+2)\alpha}{2k(4k+2)} \rho \left[1 - \left(\frac{\rho}{R_2} \right)^{4k} \right] + \\ &+ \sum_{k=1}^{\infty} \frac{\cos(4k-2)\psi \cos(4k-2)\alpha}{2k(4k-2)} \rho \left[1 - \left(\frac{R_1}{\rho} \right)^{4k} \right] \end{aligned} \quad (30)$$

$$\begin{aligned} \frac{B_{\psi}}{4\lambda} &= - \sin 2\psi \cos 2\alpha \rho \ln \frac{R_2}{\rho} - \sum_{k=1}^{\infty} \frac{\sin(4k+2)\psi \cos(4k+2)\alpha}{2k(4k+2)} \rho \left[1 - \left(\frac{\rho}{R_2} \right)^{4k} \right] \\ &+ \sum_{k=1}^{\infty} \frac{\sin(4k-2)\psi \cos(4k-2)\alpha}{2k(4k-2)} \rho \left[1 - \left(\frac{R_1}{\rho} \right)^{4k} \right] \end{aligned} \quad (31)$$

Distribution : (open)

MPS Scientific Staff

Appendix

Let $Z = \cos \theta + i \sin \theta$ and x a real number < 1 ; then

$$Z \sum_{k=1}^{\eta} (xZ)^{k-1} = Z \frac{1-(xZ)^{\eta}}{1-xZ}$$

as the sum is a geometrical series, If $\eta \rightarrow \infty$, then $\lim_{\eta \rightarrow \infty} (xZ)^{\eta} = 0$ and

$$\lim_{\eta \rightarrow \infty} \sum_{k=1}^{\eta} x^{k-1} Z^k = \sum_{k=1}^{\infty} x^{k-1} Z^k = \frac{Z}{1-xZ} = \frac{\cos \theta + i \sin \theta}{1-x \cos \theta - ix \sin \theta}$$

Using De Moivre's formula for the left side expression and making the denominator real on the right hand, we obtain, after separating the real and imaginary parts :

$$\sum_{k=1}^{\infty} x^{k-1} \cos K \theta = \frac{\cos \theta - x}{1-2x \cos \theta + x^2}$$

$$\sum_{k=1}^{\infty} x^{k-1} \sin K \theta = \frac{\sin \theta}{1-2x \cos \theta + x^2}$$

Integrating the first expression one obtains :

$$\sum_{k=1}^{\infty} \frac{x^k \cos K \theta}{K} = -\frac{1}{2} \log (1-2x \cos \theta + x^2)$$